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THIN VISCOUS ELLIPTICAL ACCRETION DISCS WITH ORBITS SHARING A COMMON LONGITUDE OF PERIASTRON
III. NUMERICAL EVALUATIONS OF THE VALIDITY DOMAIN OF THE SOLUTIONS

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Abstract

We have investigated the validity domain of the dynamical equation which defines the structure of a two-dimensional elliptical accretion disc model of Lyubarskij et al. [9]. Only cases with integer powers in the viscosity law $\eta \sim \Sigma^n$ are considered, namely n = -1, 0, +1, +2 and +3 (η is the viscosity coefficient, Σ is the disc surface density). This approach is adopted in view of the fact that the analytical expressions for the dynamical equation for these particular values of n are already derived in an earlier paper [7]. As a mathematical problem, we have to solve a second order ordinary differential equation with initial conditions – two arbitrary constants e_0 (the value of the eccentricity) and its derivative $\dot{e}_0$ for a given fixed value of the focal parameter p_0 of a selected elliptical trajectory. In the present paper we have chosen the following grid of values: $e_0 = 0.00$, $+0.20$ and $+0.50$; $\dot{e}_0$ varies by step 0.01 accordingly from –1.00 to +1.00, from –0.80 to +1.20 and from –0.50 to +1.50. The independent variable u in the dynamical equation is defined as a logarithm of the focal parameter p of the elliptical particle trajectories, i.e., $u \equiv \ln p$. Respectively, $e = e(u\,;\, e_0,\, \dot{e}_0,\, n)$ and $\dot{e} = \dot{e}(u\,;\, e_0,\, \dot{e}_0,\, n)$. By the definition of the problem, each eccentricity e must be a real function and from physical reasons the inequalities $|\,e(u)\,| < 1$, $|\dot{e}(u)| < 1$ and $|\,e(u) - \dot{e}(u)\,| < 1$ must be satisfied for every 3-tuple of parameters (e_0, $\dot{e}_0$, n). Then the dynamical equation is solved by means of numerical methods and the range of variation of u where the above restrictions are satisfied is found out. For each of the 15 combinations (n, e_0) the permitted range of variation of u as a function of $\dot{e}_0$ is presented graphically.

Introduction

Accretion flows in astrophysics are widely occurring phenomena. They are established for certain to exist around protostars, accreting compact objects in binary stellar systems, and also around supermassive black holes at the central regions of galaxies. In other words, configurations taking the form of flattened discs are commonly observed in astronomy. When accretion discs are highly flattened, the material in these objects typically orbits in circles, such that the centrifugal force to a first approximation balances the gravitational attracting force. The latter is often due to some external gravitational potential into which the matter has fallen, but can also (in part or in whole) be caused by the self-gravity of the material itself. Standard accretion disc theory describes the viscous evolution of flat circular discs around a single object (ordinary star, white dwarf, neutron star or black hole). However, accretion discs are often distorted in various ways. A disc around a single star may be composed of non-coplanar or elliptical orbits of its particles, resulting in a warped or eccentric disc, respectively. In addition, discs in binary and protoplanetary systems are tidally distorted by the presence of the orbiting companion. To describe of these phenomena, during the recent decade semi-analytical methods are developed and applied . They bring us nearer to the solution of these essentially three-dimensional, nonlinear problems [1]. Consequently, the simplest disc models are those whose disc elements follow circular orbits around a central potential in the same plane. But due to interactions with external forces, the discs are often neither coplanar nor circular.

It should be noted as well that a certain amount of work has also been done on eccentric and non-planar particle discs around planets. In this connection, it may be noted that the situation with regard to fluid discs is much less clear. Narrow planetary rings are often observed to be non-circular, exhibiting sometimes $m = 1$ (eccentric ring) or $m = 2$ distortions. This phenomenon can occur because the action of viscosity through the shear can destabilize non-axisymmetric density waves (viscous overstability). Another possible cause is parametric instability caused by a tidal interaction with a companion satellite. Such situations may also occur in larger in larger scale discs. There, viscosity is again capable to excite eccentricity rather than to cause it to decay. It makes sense to postulate that a combination of tides and parametric instability is causing disc eccentricity in dwarf novae stars, inspite of the circumstance that the conditions required for these instabilities (they depend on the viscosity law) are not understood sufficiently well at present. It is worth to note that an opportunity for cross-comparison between planetary ring dynamics and gaseous accretion discs is a promising perspective for better understanding of the conditions required for viscosity-driven non-axisymmetric distortions to occur in discs. The preliminary results obtained at this stage are not essentially new but serve as an important test for disc models. This stimulated the collaboration between astronomers whose aim is studying the properties of eccentric discs in binary stellar systems.

Discs can become non-planar and/or eccentric due to interaction with the external constraints. For example, in the protostar field, most stars have binary companions and they form in crowded regions. Thus, protostellar discs are likely to be subject to interactions with companions or even with passing interlopers. It is also possible that when two stars come together to form a binary star, there can be an exterior protostellar disc surrounding the binary system as a whole. Dynamically, the impact of the stars affects the discs, but there is also a response reaction of the disc on the orbits of the stars themselves. It was shown, however, that even for the most compact star-forming environments known, disc penetrating encounters leading to capture of the star would be rare, with the probability of such an encounter being approximately 10 % over a disc's lifetime. But if the mean free-path in the star-forming core is sufficiently small so as to allow close encounters between young stellar objects on a time scale that is shorter than the lifetime of a protostellar disc, then the tidal interaction between a secondary object and the disc of the primary may nonetheless be a dynamically significant event in the disc's history.

S. Goodchild and G. Ogilvie have applied an eccentric accretion disc theory in a simplified form to the case of superhumps in SU Ursae Majoris cataclysmic variables and other similar systems [2]. In their model the disc contains 3:1 Lindblad resonance and it is believed that this resonance leads to growth of eccentricity and a modulation in the light curves due to the interaction of precessing eccentric discs with tidal viscous damping of eccentricity. These authors first worked out the theory in the simple case of a narrow ring and they arrived at the conclusion that the eccentricity distribution is locally suppressed by the presence of the resonance, creating a dip in the eccentricity at the resonant radius. Application of this theory to the case of superhumps in close binary stars confirms observationally this conclusion. It also gives possibilities for more accurate expressions for the precession rates of the discs than has been previously accomplished by means of simple dynamical estimates.

Time-resolved spectroscopy of the nova-like variable UU Aquarii is analyzed with eclipse mapping techniques to obtain spatially resolved spectra of its accretion disc and gas stream as a function of the distance from the disc centre in the spectral range $3600 \div 6900$ Å The interpretation of the eclipse maps suggests that the asymmetric structure in the outer disc, which was previously identified as a bright spot, may be a signature of an elliptical disc similar to those which probably are present in SU Ursae Majoris stars during outbursts [3]. The possibility that protoplanetary gaseous discs are dynamically unstable to axisymmetric and non-axisymmetric gravity perturbations is studied by E. Griv [4]. This investigation includes also perturbations that are produced by spontaneous disturbances. The analytical treatment of these phenomena (with characteristic scales larger than the vertical scale-height) is realized by using a local Wentzel-Kramers-Brillouin (WKB) approach. This paper reveals an interesting connection between non-axisymmetric accretion discs and their clumpy structures, which are gravitationally bound. The latter can collapse to become giant planets. There is an important feature of the planetary formation process. Namely, gravitationally unstable non-axisymmetric (e.g., spiral) perturbations can effectively transport both the angular momentum and the mass in a spatially inhomogeneous disc. Another investigation of the action of gravitational perturbers in thin cold astrophysical discs is carried out in paper [5]. Two types of density structures are found, depending on the mass of the perturbing body and on the amount of momentum transport in the disc. A gap around the whole circumference of the disc is opened if the perturber is more massive than a certain threshold. In the opposite case, a local S-shaped density modulation is generated (the so called "propeller"). It was found that the large-scale appearance of the "propellers" does not depend on the details of the scattering process but mainly on the effective strength of gravity perturbations. The solutions of the problem show that the characteristic spatial extentions of the structures depend on the mass of the perturber and the viscosity of the disc. This conclusion is in agreement with the already known result of the theory that the tidal torques exerted by a binary system on an accompanying disc may limit its spatial scope. Examples of such occurrences are an accreting protoplanet that is embedded in the nascent disc and which may clear an annulus about its orbital path, or a massive binary system which may maintain a cleared cavity within a circumbinary disc. The disc response may be studied as a function of binary mass ratio, separation between stars, inclination of the disc with respect to the orbital plane and the disc thickness. Theoretical considerations demonstrated that differentially rotating discs can precess approximately as rigid bodies provided the precessional timescale is sufficiently long when compared with the sound-crossing timescale for the disc. For self-gravitating disc models it was established that the usual condition for instability to non-axisymmetric perturbations is insufficient to ensure fragmentation and a higher level of instability, than is usually considered, appears to be required.

Numerical and analytical studies of accretion discs are motivated through reference to recent observational and theoretical results. In particular, in this respect the considerations connected with the new observational investigations on binarity among young stars should be mentioned. The analysis of nearby star-forming regions has revealed that the young stellar objects have a binary frequency in excess of that found for field stars. Roughly half of the current sample of these binary stars are associated with optically thick, geometrically thin, circumstellar accretion discs. It was found that the peak in frequency distribution over binary separation occurs at about 30 a.u., which is less than the typical disc sizes, which are about 100 a.u. These conclusions are based on the observational studies of nearby star-forming regions. Rotation of the accretion disc matter provides centrifugal support within a preferred plane which gives a geometrically thin circumbinary/circumstellar disc that is pressure supported perpendicular to that plane. In an early epoch the surrounding envelope of matter is cleared through the action of the stellar wind, giving exposed young stars/star and circumbinary/circumstellar disc, respectively.

Non-planar (or warped) accretion discs might be set up by the tidal forcing due to an inclined-orbit binary. Evidence of the non-planarity of binary orbits and disc planes is provided by the low inclinations of gas giants in the Solar system. Further, a small asymmetry in the inner regions of the dusty disc associated with β Pictoris has been interpreted as a warp generated by possibly a Jupiter-mass body on an inclined orbit embedded into the interior to the disc. Asymmetries found on the scale of the disc in this binary system might also have been caused by a stellar encounter.

Physical restrictions and features of the considered accretion disc model

In binary stellar systems the disc matter is continuously replenished by the donating companion through the mass transfer stream, which strikes the outer parts of the disc and causes a "hot spot" event. The latter is not expected to significantly affect the accretion disc dynamics. But if the rate of mass transfer is insufficient to maintain the disc temperature above the ionization temperature of hydrogen (i.e., 6000 K), then the disc can switch from an optically thin state to an optically thick state. This leads to thermal instability and outburst behaviour when the accretion rate may increase by two orders of magnitude and the disc may expand to fill the tidal radius of the compact object on time scales of weeks to months. In our further considerations we fully exclude such an obvious non-stationarity and concentrate only on the stationary accretion discs. It is also preferable to exclude explicitly the detailed treatment of the thermal processes because the emission processes are always found to be specific to particular astronomical objects, and give little opportunity, if any, for cross-field fertilization. This remark is essential to be made, because our model of the accretion disc is not so detailed to be compared immediately with the observations. However, attention must be paid to these parts of the disc where the energy is dissipated as this can have feedback into the disc structure and thus affect the angular momentum transport.

Although astrophysical discs come in a variety of forms and have a wide range of length scales changing from planetary to galactic sizes, there can be a lot of common ground between them because of similar underlying dynamical properties. For this reason, research problems in the area are grouped together according to similarity of dynamical properties, rather then under astronomical field. Consequently, the investigation efforts may include researches in the general areas of hydrodynamics, magnetohydrodynamics, stability of self-gravitating differentially rotating systems and the theory of turbulent dissipative fluid systems as they relate to astrophysical discs. Both analytic approaches and numerical simulations are to be involved. In the present study we shall use analytical results about elliptical accretion discs obtained for particular samples of the viscosity law. These are already published in an earlier series of papers [6], [7], and [8], where the expressions of the dynamical equation (governing the structure of the disc) are given in explicit form. Here we continue the investigation of their properties. More specially, we concentrate our attention on the determination of the domain where the solutions are not only mathematically well defined, but also have reasonable physical meanings. Because of the complexity of the analytical expressions, the methods for solving the equations are purely numerical.

Another simplification of the accretion disc model considered by us should be mentioned. In fact, this is the elliptical disc first investigated by Lyubarskij et al. [9], where the internal torque is derived consistently from the basic fluid-dynamical equations within the context of the α-theory. The latter means that the assumption that effective dynamic viscosity is proportional to local pressure is valid for the entire disc. But large-scale electromagnetic fields are not included into consideration. This situation may not always correspond to the real objects in nature. Gaseous discs are observed around compact components in interacting binary stellar systems and around very young stars, which are still in the process of forming. They also occur around massive black holes in active galactic nuclei. Such discs consist of highly ionized plasma, therefore in them magnetic, as well as purely hydrodynamic processes, have to be taken into account. The effect of the magnetic field of surface strength $B \approx 1$ T (over the surface of the secondary star) on the accretion disc structure is examined by Pearson [10]. The theory of precessing discs and resonant orbits is generalized to encompass resonances of higher order than 3:2 and is shown to retain consistency with the new mass ratio of the star AM Canum Venaticorum. Such investigations show that neglecting the magnetic field in the accretion disc model of Lyubarskij et al. [9] considered in the present paper may have additional difficulties if we try directly to compare it to the observational data. As mentioned also by Shalybkov and Rüdiger [11], the presence of an imposed vertical magnetic field may drastically influence the structure of thin accretion discs. If the field is sufficiently strong, the rotation law can depart from the Keplerian one. Of course, this situation is very different from that which is described by the dynamical equation derived in [9].

Neglecting the three-dimensional structure of the accretion flows may have important consequences. When the latter cannot cool via emission of radiation, they become vertically thick and nearly spherical. Thus, they are intrinsically multidimensional. Let H be the half-thickness of the disc at distance R from the center of the compact star. If the accretion disc is thick, so that $H \approx R$ and the Mach number $M \approx 1$, then all the thin disc approximations break down simultaneously. In particular, we may no longer neglect the radial pressure gradient, nor may we assume that the temperature gradients $\partial T/\partial z >> \partial T/\partial R$, where z is a coordinate perpendicular to the central disc plane. We also cannot neglect the heat flow in the radial direction. Thus, the accretion disc is no longer Keplerian, and the splitting of the solution of the disc structure into a radial part and a part perpendicular to the disc plane is no longer valid. The above remarks concerning the three-dimensional nature of accretion flows are valid not only for gaseous matter but also for dust composed discs. Such accretion discs exist around protostellar objects and are almost certainly self-gravitating during the early part of their lives, although at later times the central star dominates the gravitational potential. Albeit a dust shell may be located at quite a distance from the central binary star (up to about 1000 a.u.), it can be viewed as a remnant of a tidally-disrupted accretion disc. This seems to indicate that the disc and the orbits of the dust particles are not necessarily coplanar in the pre-main sequence stellar systems. Similar difficulties in simulations of accretion flows arise when clumping of the matter has to be taken into account. The formation of collapsed objects during the clumping of the material in a circumstellar discs is very hard to model numerically because of the huge range of spatial scales which are involved in the problem. Numerical experiments also demonstrate that failure to resolve the vertical structure of discs leads to greate errors in the numerical solution for the evolution of the entire physical system. While the two-dimensional simulations allow much higher linear resolution in the two dimensions which they actually model, they do so at the cost of requiring that some assumptions must be made about the character of the system and its behavior in the third dimension. In the elliptical disc model of Lyubarskij et al. [9] investigated by us, where in fact an analytical description is used, the two-dimensional approach ensures that the final dynamical equation is a second order ordinary differential equation. This simplification of the problem is accompanied by the assumption of existence of local hydrostatic equilibrium at each point of the disc in the vertical z-direction.

But even when the third dimension is "eliminated" by means of appropriate assumptions about the vertical structure of the disc (i.e., it is not included explicitly in the dynamical equation), there are, of course, other inhomogeneities in the two-dimensional disc plane which complicate the analytical and numerical analysis. In the elliptical disc of Lyubarskij et al. [9] one such global inhomogeneity, namely the disc ellipticity, is incorporated into the description of disc equations. In that model global structures like spirals not taken into account. For example, the latter can arise due to shock waves, but they are not included into the model [9] because the quantities (e.g., the determinant of the metric, the introduced auxiliary functions, etc.) which are used, become divergent in that case. Consequently, the derived formulae in [9] are not appropriate to describe these complicated situations despite their closer similarity to the reality. This circumstance is, unfortunately, an additional reason obstructing the comparison "theoretical model – observations". The same is the case with the conclusion that the spirals contribute by about 16 and 30 per cents to the total line flux, respectively, for the He and CN4650 emission lines [12]. It is interesting to note that the comparison of the Doppler and eclipse maps for IP Pegasi reveals that the Keplerian velocities derived from the radial position of the spiral shocks are systematically larger than those inferred from the Doppler tomography. This indicates that the gas in the spiral shocks has sub-Keplerian velocities – another distinction from the model considered in the present paper.

The difficulties concerning the description of accretion disc structure become even more trouble some when the fragmentation into local density blobs must be included. These problems are by necessity solved by means of numerical methods. As computers become more powerful, the studies previously limited only to local patches of the accretion flows may be now expanded into global studies that encompass the entire disc. But in this area of investigations much must be done to overcome various obstacles. In particular, from the numerical simulations it follows that an error in the cooling rate as large as a factor of two will be sufficient to suppress or enhance the fragmentation in a simulation, which would otherwise follow a very different evolutionary path [13]. An error of factor of two in the cooling rate will be equivalent to an error of ≈ 20 % in the photosphere temperature, due to the T_{eff}^4 dependence of the cooling on the temperature.

It would be stressed that a description of a real physical system (more specially, when it must be compared with the observational data) requires not only analytically or/and numerically valid simulations, but also relevant initial conditions and a correct and complete physical model. As mentioned by A. Nelson: "...Numerically valid simulations will be interesting only to the extent that they model real systems with real physics. The important point to take from these discrepant results is that not only do questions of numerical validity remain to be addressed, but also the physical models." [13]. Definitely, such conclusions remain valid also for analytical treatment of the problems, which concern the dynamics and structure of accretion discs.

Concrete values of the parameters and numerical evaluations of the validity domains for disc dynamical equation solutions

Briefly summarizing the results already obtained in the previous papers [6] and [7] about the accretion disc model worked out by Lyubarskij et al. [9], we shall remind that the viscosity coefficient η is related to the disc surface density Σ via the relation $\eta = \beta\, \Sigma^n$. Here β is a constant and the power n may, generally speaking, vary continuously within an interval with upper and lower boundaries determined on the basis of physical reasons, corresponding to the accepted accretion disc model. It is worth to note that the description of the disc given by Lyubarskij et al. [9] is an extension of the standard α - disc theory of Shakura and Sunyaev [14] to the case of accretion flows with elliptical orbits of the particles around the central star. In view of the fact that we have obtained earlier explicit forms of the dynamical equation only for five values of the power n [7], we limit our results in this paper also only to these particular values of n. Namely, our investigation of the validity domains is for the following fixed values:

$n = -1, 0, +1, +2$ and $+3$.

The dynamical equation determining the structure of the considered stationary accretion disc [9] is a second order differential equation. Hence, its solutions depend on two arbitrary integration constants. It is appropriate to choose as such constants the values of the eccentricity e and its derivative $\dot{e}$ at a given value u_0

of the argument. Each ellipse is described by its eccentricity e and focal parameter p. According to Lyubarskij et al. [9], it is plausible to make a transition to a new variable $u \equiv \ln p$. Having in mind that for a circular orbit p is equal to the radius r of the orbit, the above substitution simply means that we have introduced a logarithmic scale to characterize the spatial size of the elliptical orbits of disc particles. Lyubarskij et al. [9] allow variations of u in the interval from $u = -3.0$ (innermost orbits of the disc) to $u = +3.0$ (outermost orbits of the disc). We shall keep these limits in the present study. The intermediate value $u = 0.0$ corresponds to $p = +1.0$. We have accepted this value as a point where the initial conditions are fixed also. Therefore, we introduce the following notations: $p_0 = +1.0$, $u_0 \equiv \ln p_0 = 0.0$, $e_0 \equiv e(u_0)$ and $\dot{e}_0 \equiv \partial e(u)/\partial u\ |_{u=u0}$. In the next we have selected to investigate the solutions of the dynamical equation for three different values of e_0 for every fixed value of n; namely, $e_0 = 0.0$, $+0.20$ and $+0.50$. This choice ensures 15 combinations corresponding to the above representative discrete sample of values for n and e_0. In connection with this lattice of values, an important remark must be made on the signs of the eccentricity $e(u)$ and its derivative $\dot{e}(u) \equiv \partial e(u)/\partial u$ (in particular, this concerns e_0 and $\dot{e}_0$). By its way of deriving and by the formulation of the task itself (all particles' trajectories are postulated to be ellipses which apse lines are in line with each other), the dynamical equation has a symmetry in respect to the change of the sign of the eccentricity $e(u)$. The latter operation automatically does so with the signs of the derivatives $\dot{e}(u)$ and $\ddot{e}(u)$. In other words, if the periastron of an ellipse lies on the positive part of the abscissa, then its mirror ellipse with a periastron on the negative part is also a solution of the dynamical equation. This symmetry property can easily be seen from the explicit forms of the considered equation given for integer values of the power n [7]. The just above-mentioned characteristic must not be misleadingly identified with the possibility that some ellipses may have positive eccentricities (to say, in the inner parts of the accretion disc) and negative eccentricities (to say, in the outer parts of the same disc) and vice versa. Of course, if the accretion disc undergoes a mirror transformation as a whole (i.e., all ellipses change the signs of the eccentricities and their derivatives), formally the geometry of the accretion flow changes but physically the situation is not different. For this reason, we shall not include in our examples negative values of the constant of integration e_0. Negative initial values e_0, combined with negative/positive initial values of $\dot{e}_0$ will correspond to positive initial values of e_0 (with the same absolute values) combined with positive/negative values of $\dot{e}_0$ (again with the same absolute values as the former $\dot{e}_0$). Concluding this remark upon the signs of the eccentricities, it is worth to note that the notion "negative eccentricity" was introduced by Lyubarskij et al. [9] simply to express analytically (and graphically) in a more compact form the global changes of eccentricity through the entire disc. "Negative eccentricity" simply means that the periastron of the ellipse (which really has a positive eccentricity with the same absolute value) lies on the negative part of the abscissa. Such representation is useful when there is a transition through the disc from one type of ellipses (with $e(u) > 0$) to another type of ellipses (with $e(u) < 0$) and vice versa. Only for generality, we mention that it is also helpful to use eccentricities which have complex values [15]. But this notion makes sense when more general models of accretion discs (where the simplification that the orbits share a common longitude of periastron is not already valid) are investigated. For our purposes such a complication is only an unnecessary "exotic".

Selecting the values of the parameters n and e_0, we are ready for each couple (n,e_0) to vary the second integration constant namely $\dot{e}_0$. This variation is performed with a much denser grid of points. The step of increasing of $\dot{e}_0$ is chosen to be 0.01 and the grid covers uniformly an interval of length 2.00. The initial (lower) and, correspondingly, the final (upper) boundaries of the interval are determined on the basis of the following reasons. A priori (by the definition of the problem) the eccentricity of the orbit $e(u)$ may change continuously from -1.0 (degenerate ellipse whose periastron has negative abscissa) to $+1.0$ (degenerate ellipse whose periastron has positive abscissa). The intermediate case $e(u) = 0$ corresponds to a circular orbit. In the original paper of Lyubarskij et al. [9] three auxiliary functions $Y(e,\dot{e},n;\varphi)$, $Z(e,\dot{e},n;\varphi)$ and $W(e,\dot{e},n;\varphi)$ are introduced. They are averaged over the azimuthal angle φ by integrating over φ from 0 to 2π. Then the coefficients of the dynamical equation of the elliptical accretion disc are expressed through these functions and their first partial derivatives with respect to e and $\dot{e}$. Note that in contrast to paper [9], we use the overdot symbol ($\dot{}$) to designate differentiation with respect to the variable u instead of differentiation with respect to time t. Such a change is not confusing because we consider only stationary accretion flows ([6], [7] and [8]). As pointed out in [6], an explicit computing of the angle averaged coefficients of the dynamical equation in the case of integer n ($n = -1, 0, +1, +2, +3$) requires analytical evaluations of seven integrals over the angle φ, which we denote as $\mathbf{I}_0(e,\dot{e},n),\ldots,$ $\mathbf{I}_{0\_}(e,\dot{e},n)$. Their integrands contain in the denominators multipliers of the following type: $[1+(e-\dot{e})\cos\varphi]^{n+1}$ and $[1+(e-\dot{e})\cos\varphi]^{n+2}$ (see equations (5) ÷ (7) in paper [6]). When $[e(u)-\dot{e}(u)]\cos\varphi = -1$, singularities arise which cannot be cancelled out by means of the other multipliers in the integrands. To avoid this peculiar situation, it is necessary to impose the condition $|e(u)-\dot{e}(u)| < 1$, which ensures that the integration over the azimuthal angle φ will not lead to divergences of the integrals

$\mathbf{I}_0(e,\dot{e},n),\ldots,\mathbf{I}_0(e,\dot{e},n)$. If we do so, the same inequality must hold also for the particular values of the eccentricity $e_0 \equiv e(u_0)$ and its derivative $\dot{e}_0 \equiv \dot{e}(u_0)$| $e_0 - \dot{e}_0$| < 1, i.e., $-1 < e_0 - \dot{e}_0 < +1$. Having in mind that the initial value e_0 ($e_0 = 0.00$, $+0.20$ and $+0.50$ in our computations) is already fixed when we begin to vary the second initial value $\dot{e}_0$, we can write: $-1 - e_0 < -\dot{e} < 1 - e_0$, or multiplying by -1, we have: $-1 + e_0 < \dot{e} < 1 + e_0$. Consequently, for our particular choices of e_0, we obtain the following intervals for variation of the parameter $\dot{e}_0$:

1. $e_0 = 0.00$; then $-1.00 < \dot{e}_0 < +1.00$;
2. $e_0 = +0.20$; then $-0.80 < \dot{e}_0 < +1.20$;
3. $e_0 = +0.50$; then $-0.50 < \dot{e}_0 < +1.50$.

We emphasize that the above restrictions on $\dot{e}_0$ ensure that the divergences generated by the averaging over φ (i.e., averaging over the full elliptical trajectory length, corresponding to an arbitrary fixed value of the focal parameter p (or $u \equiv \ ln\, p$) are avoided. But the elimination of the angular coordinate φ from the following sequence of mathematical manipulations and, possibly, other physical simplifications, does not guarantee that new divergences would not arise. For example, we may have expressions whose denominators include quantities like the multipliers $e(u)$, $\dot{e}(u)$, $1 - [e(u) - \dot{e}(u)]$, etc., taking zero values for some values of the independent variable $u \equiv ln\, p$. Fortunately, these complications, in fact, do not generate unresolvable difficulties. By means of the L'Hospital's theorem (namely, that which resolves indeterminations of the type 0/0), it may be seen that analytically the divergences, caused by the nullifications of the denominators, cancel out by the nullifications of the corresponding nominators. As a result, the expressions are not divergent. However, troubles may arise at these singular points, when the expressions are evaluated with the help of numerical methods, because of the finite step of the lattice. This circumstance has to be taken into account when the numerical approach is used.

After the choice of the three parameters n, e_0 and $\dot{e}_0$ is already done, we are in a position to vary the parameter u (i.e., the focal parameter p; $u \equiv ln\, p$). Confining ourselves to the directing paper of Lyubarskij et al. [9], we limit the range of variation of u from -3.0 to $+3.0$. In this paper we have varied u by step 0.001, which enables us to obtain more than enough dense grid of solutions of the dynamical equation. The latter is a second order ordinary differential equation with already fixed initial conditions e_0 and $\dot{e}_0$, and we have solved it numerically using the standard techniques for numerical solving of such equations. As discussed above, from physical point of view we have to select only these solutions for the eccentricity $e(u)$ which satisfy simultaneously the following three inequalities $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and

$|e(u) - \dot{e}(u)| < 1$. Having a grid of 6000 solutions for different values of u (reminding that n, e_0 and $\dot{e}_0$ are fixed earlier), it is not difficult to establish the boundaries within which the above conditions are satisfied. Using a linear interpolation, the precision of the determinations is about $0.0001 \div 0.00001$. This accuracy is enough, all the more we express the results graphically on Figures $1 \div 5$.

Summarizing all the parametrization of the task, which we have applied, we have chosen 15 ordered tuples (n, e_0) (n = -1, 0, +1, +2, +3 ; e_0 = 0.00, + 0.20, + 0.50). For each of them we have drawn two-dimensional plots in the plane ($\dot{e}_0$, u), giving descriptive representations of the domains of validity of the dynamical equation. The regions shaded by vertical lines on figures $1 \div 5$ denote the physically meaningless areas of solutions where the above mentioned three inequalities $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ are not satisfied. The remaining unshaded ("white") regions are the domains of validity we are seeking for in the present paper. The boarders between the "permitted" and "forbidden" zones are delimited by thick curved lines.

Discussion and concluding remarks

Each of the graphics represented in Figures $1 \div 5$ (for preliminary fixed values of n and e_0) gives a visual illustration about the set of all possible solutions for the entire (global) accretion disc. For a given viscosity law $\eta = \beta \Sigma^n$ (n is fixed ; the value of the constant β is meaningless for our consideration) and selected initial conditions e_0 and $\dot{e}_0$, the focal parameter of the elliptical trajectories of disc particles must obey the restrictions imposed according to Figures $1 \div 5$. This ensures that the eccentricities $e(u)$ and their derivatives $\dot{e}(u)$ will have physically reasonable values. Evidently, the freedom remains that between trajectories samples with different eccentricities may be present, varying from the inner to the middle and to the outer parts of the concrete accretion disc model. In principle, even if the dynamical equation is not explicitly solved, graphics like these in Figures $1 \div 5$ give to some extent an idea of what properties of the solutions $e(u)$ may be generally expected. In turn, this information will be useful for more suitable choices of the initial conditions e_0 and $\dot{e}_0$ which guarantee the uniqueness of the solution to the accretion flow.

It is evident from Figures 1a $\div$ 5a that when we choose the initial conditions to be given on a circle (i.e., $e_0 = 0$; $u = 0$; $p = 1$), then the forbidden zones for u are preferably situated in the upper half of the plane ($\dot{e}_0$, u). An exception of this property is the case $n = 0$, corresponding to the absence of functional dependence between the viscosity coefficient η and the disc surface density Σ. In order to represent more clearly the descriptive results from Figures $1 \div 5$, let us return to the definitions of the quantities u, focal parameter p and the eccentricity $e(u) \equiv e(p)$ [9]. If a and b are the major and minor semiaxes, respectively, of a single elliptical trajectory, then its eccentricity is given by $e = (1 - a^2/b^2)^{1/2}$. Taking into account that the focal parameter p of an ellipse is expressed through a and b as $p \equiv b^2/a$, we can write $e = (1 - a/p)^{1/2}$. Accepting the new variable $u \equiv ln\, p$ [9], this inversely means $p \equiv exp\, u$. There is a subtle detail. The argument of the logarithmic function must be dimensionless. So we have to express the focal parameter $p \equiv b^2/a$ as a product of two multipliers. One of them has constant value equal to unity and dimension of length – let us denote it by 1. The other multiplier is dimensionless and its value is equal to the value of b^2/a. When the definition $u \equiv ln\, p$ is introduced, just the second (dimensionless) multiplier, denoted again by the symbol p because we are interested in the quantitative nature of the focal parameter is implicitly subtended. Strictly speaking, we have to write the "dimensionally right" focal parameter as $p \equiv 1 \times exp\, u$, and hence, the quotient $a/(1 \times exp\, u)$ is thus dimensionless. Making the above note, which is caused by the omission of the dimensional multiplier 1, we are sure that this simplification of notations does not lead to any erroneous conclusions.

We also emphasize the fact that when we consider accretion discs with elliptical trajectories of the particles, both semiaxes a and b may vary for different regions of the flow. Consequently, in the general case a and b are functions of u: $a = a(u)$ and $b = b(u)$. When we write the expression for the eccentricity $e = [1 - a(u) / (1 \times exp\, u)]^{1/2}$ this does not mean that we have solved the problem at all. The semiaxis $a(u)$ is an unknown function of u. Hence, the solution of the dynamical equation is an unavoidable necessity in order to find the two-dimensional structure of the accretion disc. Having in mind that both semiaxes $a(u)$ and $b(u)$ are limited by the above functions (the disc has a finite size), it is possible to use the relation $e(u) = [1 - a(u)/(exp\, u)]^{1/2}$ (we omit the multiplier 1) to make some conclusions when we consider boundary constraints. Table 1 contains some values of $exp\, u$ when u varies within the interval [-3.0, +3.0].

Table 1

u	-3,0	-2.0	-1.0	0.0	+1.0	+2.0	+3.0
$exp\, u$	0.0498	0.1353	0.3679	1.0000	2.7183	7.3891	20.0855

It is evident from these data that *exp u* changes up to 400 times from its minimal to its maximal value in that interval. Roughly speaking, when u is "small" (i.e., u has negative values close to –3.0), the quotient $a(u)/exp\, u$ is larger in comparison to the case when u is "large" i.e., u has positive values close to +3.0). Correspondingly, in the first case the difference $1 - a(u)/exp\, u$ will be smaller then in the second case. In other words, it may be said to some naive extent, that the positive values of u suggest more elongated elliptical trajectories and the negative values of u are associated with orbits closer to circles.

Returning to Figures 1a ÷ 5a, only in the above sense we are able to conclude that except the case $n = 0$, the other cases n = -1, +1, +2 and +3 are more suggestive for a set of orbits which are not very elongated. Relative to the systems of boundary conditions given on "weakly" elongated ellipses ($e_0 = +0.20$; Figures 1b ÷ 5b) and on "strongly" elongated ellipses ($e_0 = +0.50$; Figures 1c ÷ 5c), this deduction does not alter essentially when n = +1, +2 and +3. Some differences appear for n = -1 and 0. There are practically not upper limits for u in the considered ranges of variation of $\dot{e}_0$. Consequently, such accretion discs probably contain also a larger number of high eccentricity orbital trajectories of their particles.

Another interpretation of the graphics in Figures 1 ÷ 5 may be given. By the definition of the task, the eccentricity $e(u\,; e_0, \dot{e}_0, n)$ must be a real function of u for all allowed values of the parameters e_0, $\dot{e}_0$ and n, This requirement was already taken into account when the dynamical equation was solved numerically in order to determine its validity range. Therefore, the permitted (unshaded) regions in Figures 1 ÷ 5 already take into consideration this restriction. This means that $1 - a(u)/p \geq 0$, or $a(u) \leq exp\, u$. Taking a logarithm from the latter inequality, we obtain $ln\, a(u) \leq u$. Consequently, it is possible to give a new interpretation of the results presented in figures 1 ÷ 5. They may be considered as restrictions on the major semiaxes of the particle trajectories. Such limitations are imposed under definite values of the parameters n, e_0 and $\dot{e}_0$.

In conclusion, it should be mentioned again that despite the use of the analytical forms of the dynamical equation, it was solved by means of numerical methods. For this reason we do not give denser grid with respect to the parameter e_0. Moreover, we have not available analytical expressions for that equation in the cases where the power n is not integer. Nevertheless, Figures 1 ÷ 5 provide a general picture of the validity range of the dynamical equation.

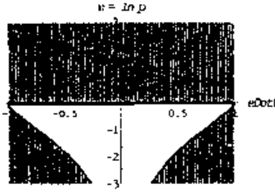

Fig. 1a. Case n = -1 ;
e_0 = 0.00

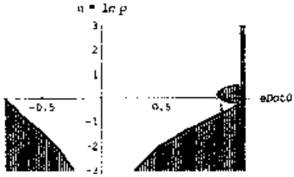

Fig. 1b. Case n = -1 ;
e_0 = + 0.20

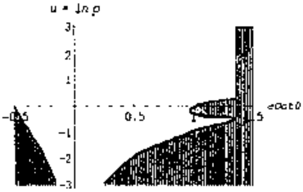

Fig. 1c. Case n = -1 ;
e_0 = +0.50

Fig. 1. Permitted range of variation of $u \equiv \ln p$ (p is the focal parameter of the elliptical trajectory of a single disc particle) as a function of e_0. The vertically shaded regions are forbidden zones, which cannot be occupied by the focal parameters of the disc orbits. The power in the viscosity law $\eta \sim \Sigma^n$ is $n = -1$; e_0 is an initial value of the eccentricity (i.e., an arbitrary integration constant), which we choose to take three fixed values.

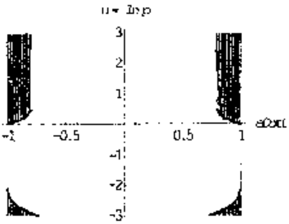

Fig. 2a. Case n = 0 ;
e_0 = 0.00

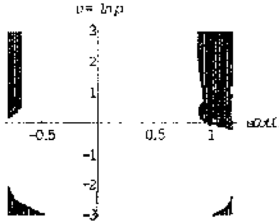

Fig. 2b. Case n = 0 ;
e_0 = +0.20

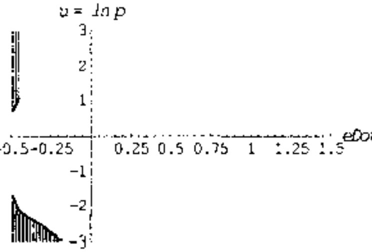

Fig. 2c. Case n = 0 ; e_0 = +0.50

Fig. 2. The same as in Fig. 1., but the power in the viscosity law $\eta \sim \Sigma^n$ is equal to $n = 0$.

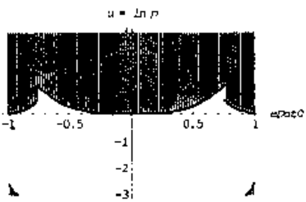

Fig. 3a. Case n = + 1 ;
e_0=0.00

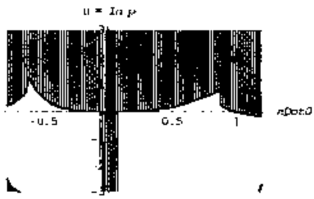

Fig. 3b. Case n = +1 ;
e_0=+0.20

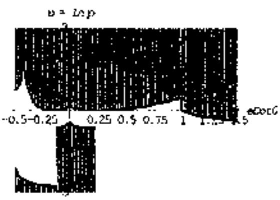

Fig. 3c. Case n = +1 ;
e_0=+0.50

Fig. 3. The same as in Fig. 1., but the power in the viscosity law $\eta \sim \Sigma^n$ is equal to $n = +1$.

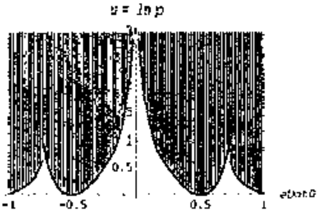

Fig. 4a. Case n = +2 ;
e_0 = *0.00*

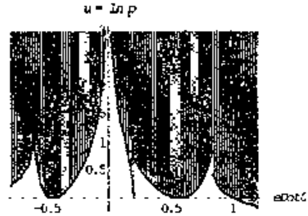

Fig. 4b. Case n = +2 ;
e_0 = *+0.20*

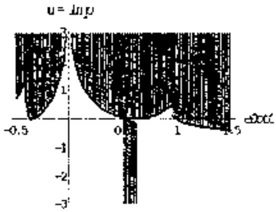

Fig. 4c. Case n = +2 ;
e_0 = *+0.50*

Fig. 4. The same as in Fig. 1., but the power in the viscosity law $\eta \sim \Sigma^n$ is equal to n = +2 .

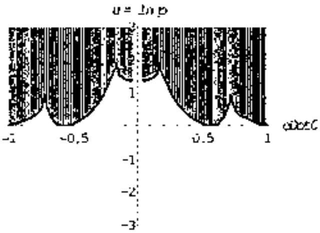

Fig. 5a. Case n = +3 ;
e_0=*0.00*

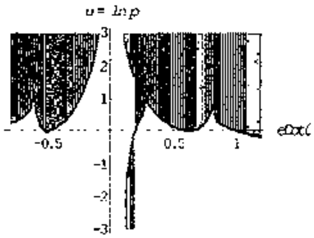

Fig. 5b. Case n = +3 ;
e_0=*+0.20*

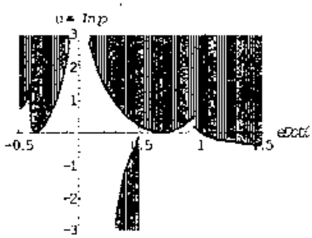

Fig. 5c. Case n = +3 ;
e_0=*+0.50*

Fig. 5. The same as in Fig. 1., but the power in the viscosity law $\eta \sim \Sigma^n$ is equal to n = +3 .

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ТЪНКИ ВИСКОЗНИ ЕЛИПТИЧНИ АКРЕЦИОННИ ДИСКОВЕ С ОРБИТИ ИМАЩИ ОБЩА ДЪЛЖИНА НА ПЕРИАСТРОНА. III. ЧИСЛЕНИ ОЦЕНКИ НА ОБЛАСТТА НА ВАЛИДНОСТ НА РЕШЕНИЯТА

Д. Димитров

Резюме

Ние сме изследвали областта на валидност на динамичното уравнение което определя структурата на един двумерен модел на елиптичен акреционен диск на Любарски и др. [9]. Разгледани са само

само случаи на целочислени степени в закона за вискозитета $\eta \sim \Sigma^n$, а именно n = -1, 0, +1, +2 и +3 (η е коефициентът на вискозитета, Σ е повърхностната плътност на диска). Този подход е възприет с оглед на факта, че аналитичните изрази за динамичното уравнение за тези частни стойности на n са вече получени в една по-ранна статия [7]. Като една математическа задача, ние трябва да решим едно обикновено диференциално уравнение от втори ред с начални условия – две произволни константи e_0 (стойността на ексцентрицитета) и неговата производна $\dot{e}_0$ за една дадена фиксирана стойност на фокалния параметър p_0 на една избрана елиптична траектория. В настоящата работа ние сме избрали следната мрежа от стойности: e_0 = 0.00, +0.20 и +0.50 ; $\dot{e}_0$ варира със стъпка 0.01 съответно от –1.00 до +1.00, от –0.80 до +1.20 и от –0.50 до +1.50. Независимата променлива u в динамичното уравнение е дефинирана като логаритъм от фокалния параметър p на елиптичните траектории на частиците, т. е. $u \equiv \ln p$. Съответно, $e = e(u\,;\, e_0,\ \dot{e}_0, n)$ и $\dot{e} = \dot{e}(u;\ e_0,\ \dot{e}_0,\ n)$. Съгласно дефинирането на задачата, всеки един ексцентрицитет e трябва да бъде реална функция и въз основа на физически съображения трябва да бъдат удовлетворени неравенствата $|e(u)| < 1$, $|\dot{e}(u)| < 1$ and $|e(u) - \dot{e}(u)| < 1$ за всяка една подредена тройка от параметри $(e_0, \dot{e}_0, n)$. Тогава динамичното уравнение е решавано с помощта на числени методи и е намирана областта на изменение на u където са удовлетворени горните ограничения. За всяка една от 15 комбинации (n, e_0) допустимата област на изменение на u като функция на $\dot{e}_0$ е представена графически.